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# THE SELFSIMILAR ASYMPTOTIC FORM OF NON-STATIONARY VORTEX FLOWS*

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The hydrodynamic reaction of a viscous incompressible fluid filling a half-space to a rotational impulse applied to its surface is studied. It is established that for a non-stationary flow which occurs in this case, a stable, selfsimilar asymptotic form exists which is independent of the form of the initial perturbation. Asymptotic expressions are obtained for the universal distribution of the meridional velocity near the surface and at infinity.

An analogue of Bernoulli's theorem is established for a class of non-stationary selfsimilar flows of an ideal fluid, and a corresponding integral of motion is obtained for the axisymmetric case.

1. Consider a class of selfsimilar motions of a viscous incompressible fluid whose velocity field is determined by the expression

$$\mathbf{v} = \sqrt{\frac{\gamma}{t}}\,\mathbf{u}\left(\frac{\mathbf{r}}{\sqrt{\gamma t}}\right) \tag{1.1}$$

Here $\mathbf{r} \in R^3$ is the radius vector, $t$ is the time and $\gamma$ is the characteristic parameter of the problem with the dimensions of circulation. Solutions of this type may describe the asymptotic stage of the reaction of a liquid medium under the action of localized dynamic perturbations.

The system of Navier-Stokes equations for the dimensionless vector function $\mathbf{u}$ will transform, taking (1.1) into account, to the form

$$(\mathbf{u}\cdot\nabla)\,\mathbf{u} - {}^1/_2\mathbf{u} - {}^1/_2\,(\mathbf{a}\cdot\nabla)\,\mathbf{u} = -\nabla p + \varepsilon\Delta\mathbf{u}$$
$$(\nabla\cdot\mathbf{u}) = 0,\quad \varepsilon = \nu/\gamma,\quad \mathbf{a} = \mathbf{r}/\sqrt{\gamma t},\quad p = tP/(\gamma\sigma)$$

(the operators $\nabla$ and $\Delta$ act on $\mathbf{a}$; $P$ is the pressure and $\sigma$ is the density of the fluid).

2. Let us first consider some general properties of the flows of type (1.1) in the limit when the viscosity becomes vanishingly small ($\nu \ll \gamma$). In this case the last term of the first equation of (1.2) can be neglected and we can rewrite this equation in a form analogous to Euler's equation in Gromeko-Lamb form

$$\boldsymbol{\omega} \times (\mathbf{u} - {}^1/_2\mathbf{a}) = -\nabla\Pi,\quad \Pi = p + {}^1/_2\mathbf{u}^2 - {}^1/_2\,(\mathbf{a}\cdot\mathbf{u}) \tag{2.1}$$

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Multiplying both sides of Eq.(2.1) scalarly by $\mathbf{u} - {}^1/_2\mathbf{a}$, we find that $\Pi = \mathrm{const}$ on the lines of the vector $\mathbf{u} - {}^1/_2\mathbf{a}$, i.e. an analogue of Bernoulli's theorem exists for the class of non-stationary motions of the fluid in question.

When the motion of the fluid is axisymmetric, we find from (2.1) in the same manner that the angular momentum $\rho u_\varphi$ ($\rho$ is the distance along the axis of symmetry) is also constant on the lines of the vector field $\mathbf{u} - {}^1/_2\mathbf{a}$. This yields, taking the analogue of the Bernoulli integral into account, the integral of Eq.(2.1): $\Pi = F(\rho u_\varphi)$ ($F$ is a function which must be determined from the boundary conditions), in the same manner as in the stationary case /1/.

3. We will illustrate the use of the representation of the velocity field in the form (1.1) by determining the reaction of a viscous incompressible fluid occupying a half-space, to the action of a rotational impulse applied to its surface. In this case the velocity field can be sought in the form (/12/,p.192)

$$v_r = rF(z,t), \quad v_\varphi = r\Phi(z,t), \quad v_z = \Psi(z,t), \quad P = P(z,t) \tag{3.1}$$

where $r, z, \varphi$ are cylindrical dimensional coordinates.

The problems discussed in /2/ are connected with the study of the dynamics of a boundary layer on a rotating disc, in the case when its angular velocity of rotation varies according to the law $\sim t^\alpha$. (The value of $\alpha = -1$ should correspond to the case discussed in the present paper). However, the final system of equations for the selfsimilar functions lacks the non-linear convective terms which, as we have found, play a major role in the dynamics of the passage to the selfsimilar stage. Below we discuss the problem in question by applying the numerical solution of the non-stationary Navier-Stokes equations to solutions of the type (3.1).

Let us transform the Navier-Stokes equations with help of the following substitution:

$$\mathbf{r} \to \mathbf{a} = \mathbf{r}/\sqrt{\gamma(t+t_0)}, \quad t \to \tau = \ln(1 + t/t_0), \quad \mathbf{v} \to \mathbf{u} = \sqrt{\gamma/(t+t_0)}\,\mathbf{v}$$

Here $t \in [0, \infty)$, and $\sqrt{\gamma t_0}$ determines the characteristic spatial scale of the initial perturbation. Eq.(1.3) remains unchanged, and we add $\partial u/\partial\tau$ to the left-hand side of Eq. (1.2). The emergence of the solution to the stage stationary with respect to $\tau$, is equivalent to the emergence of the selfsimilar asymptotic form.

We shall assume that at the initial instant the only non-zero parameter is the azimuthal velocity component $u_\varphi = \rho\Omega_0(\xi)$ ($\xi = z/\sqrt{\gamma(t+t_0)}$), and $\Omega_0(\xi)$ decreases fairly rapidly in the direction of increasing depth of the fluid (as $\xi \to \infty$). The surface is assumed to be immobile, which is true when the perturbation is sufficiently small, and tangential stresses on it are assumed to be zero. In accordance with this, we shall seek the solution in the form

$$u_\xi = -W(\xi,\tau), \quad u_\rho = {}^1/_2\partial W/\partial\xi, \quad u_\varphi = \rho\Omega(\xi,\tau), \quad p = p(\xi,\tau)$$

Here the equation of continuity is satisfied automatically. Denoting $\partial W/\partial\xi$ by $Y$, we obtain from the first equation of (1.2) a system of equations for the dimensionless functions $Y, \Omega, p$ (the integration in $\zeta$ is carried out from 0 to $\xi$)

$$\begin{aligned} \partial\Omega/\partial\tau &= (1-Y)\Omega + \left(\int Y\,d\xi + \xi/2\right)\partial\Omega/\partial\xi + \varepsilon\partial^2\Omega/\partial\xi^2 \\ \partial Y/\partial\tau &= Y - {}^1/_2Y^2 + 2\Omega^2 + \left(\int Y\,d\xi + \xi/2\right)\partial Y/\partial\xi + \varepsilon\partial^2 Y/\partial\xi^2 \\ \partial p/\partial\xi &= (\partial/\partial\tau)\int Y\,d\xi - (Y + {}^1/_2)\int Y\,d\xi - \xi Y/2 - \varepsilon\partial Y/\partial\xi \end{aligned} \tag{3.2}$$

The system of Eqs.(3.2) is supplemented by the boundary conditions which follow from the fact that the tangential stresses on the free surface are zero.

The condition $W(\xi = 0) = 0$ was used in formulating Eqs.(3.2).

The resulting boundary-value problem was solved numerically for various initial distributions $\Omega(\xi, 0) = \Omega_0(\xi)$ under the condition that $Y(\xi, 0) = 0$. An explicit scheme was used, known /3/ to be stable and reducible to the solution when the step in $\tau$ is sufficiently small. We found, as a result, that a meridional flow appears for any initial profiles $\Omega_0(\xi)$, localized near $\xi = 0$, whose profile tends asymptotically ($\tau \gg 1$) to the universal form independent of $\Omega_0(\xi)$, while the rotation disappears.

This selfsimilar solution differs from the one obtained when, following /2/, we put $\alpha = -1$) and which yields $\Omega \neq 0$ at the selfsimilar stage. The difference is related to the fact that convective transfer, which ensures the passage of vorticity in the radial direction to infinity, is disregarded in /2/.

Figs.1 and 2 show the dynamics of the passage to the selfsimilar stage for two different profiles $\Omega_0(\xi)$: $\Omega_0 = 1$ when $0 \leqslant \xi \leqslant 1$, $\Omega_0 = 0$ when $\xi > 1$ (Fig.1) and $\Omega_0 = 0{,}2\exp(-\xi^2)$ (Fig.2). The numbers on the curves correspond to the values of $\tau$.

Let us analyse in greater detail the velocity field at the asymptotic stage when $\Omega = 0$, $\partial Y/\partial\tau = 0$. In this case the equation for the velocity component $W(\xi)$ has the form (a prime denotes a derivative with respect to $\xi$)

$$\varepsilon W''' + (W + \xi/2) W'' + (1 - W'/2) W = 0 \tag{3.3}$$

Analytically, Eq.(3.3) is hardly solvable; hence, in order to analyse its solution satisfying the necessary boundary conditions we shall use the method of matched asymptotic expansions under the assumption that $\varepsilon \ll 1$ /4/.

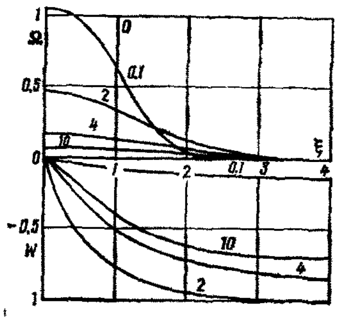


Fig.1

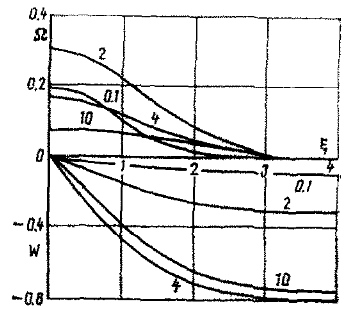


Fig.2

We can neglect the viscous term in the region $\xi \gg \sqrt{\varepsilon}$, in which case it is easy to obtain the solution in parametric form

$$W = -\frac{\xi}{2} + \frac{Ct^{3/2}}{(2-t)^{1/2}}, \quad \xi = 2C\int_0^t \frac{x^{3/2}\,dx}{(2-x)^{3/2}}, \quad Y = 2 - t \tag{3.4}$$

Here $t \in (0, 2)$ is a parameter determining the dependence of $W$ and $Y$ on $\xi$, and $C$ is a constant proportional to the velocity at infinity $W_\infty$. Passing to the limit as $t \to 2$ and eliminating the indeterminacy, we obtain

$$W_\infty = 3\pi C \tag{3.5}$$

When $\xi \to 0$, (3.4) yields the asymptotic expression

$$Y = 2 - \left(\frac{5\xi}{\sqrt{2}C}\right)^{2/5} + O(\xi^{4/5}) \tag{3.6}$$

Making the substitution $Y = 2 + N$ and linearizing (3.3), we obtain the following equation in the inner region $\xi \ll \sqrt{\varepsilon}$:

$$\varepsilon N'' + {}^5/_2 \xi N' - N = 0$$

whose solution satisfying the required condition that $N'(0) = 0$, has the form

$$N = A \cdot F(-{}^1/_5, {}^1/_2, -5\xi^2/(4\varepsilon))$$

where $F(\alpha, \beta, x)$ represents the confluent hypergeometric function. Using the asymptotic form of $F$ for $\xi \gg \sqrt{\varepsilon}$, we obtain $N \sim \xi^{2/5}$ /5/. The matching condition and solution (3.6) now yield the following relation connecting the constants $A$ and $C$:

$$A = \pi^{-1/2}\Gamma(7/10)\,(10\varepsilon/C^2)^{1/5}$$

Taking into account relation (3.5) and the fact that the substitution $W \to W_* = \varepsilon^{-1/2}W$, $\xi \to \xi_* = \varepsilon^{-1/2}\xi$ removes the parameter $\varepsilon$ from Eq.(3.3), we can conclude that $C \sim \sqrt{\varepsilon}$, and therefore $A$ is independent of $\varepsilon$. The computations carried out have shown that $C \approx 0.07\varepsilon^{1/2}$ and $A \approx 2.12$.

Thus we have found that, for the problem formulated above concerning the reaction of a semi-infinite viscous incompressible medium to the action of a rotational impulse applied to the surface, a selfsimilar stable asymptotic form is reached. The nature of the initial perturbation changes considerably in the course of this. The azimuthal velocity component vanishes in the selfsimilar stage and only the meridional flow remains, whose parameters are independent of the amplitude and form of the initial perturbation.

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# A GENERALIZED FAXEN FORMULA FOR VARIOUS FORMS OF BOUNDARY CONDITIONS*

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The method of integrating the boundary conditions in the Stokes approximation is used to obtain expressions for the forces of resistance of a spherical drop with the usual boundary conditions taking both the surface viscosity and changes in surface tension into account, as well as that of a solid sphere with boundary conditions of slippage.

Faxen established formulas for the force of resistance and momentum acting on a solid sphere with boundary conditions of adhesion, for the case when the sphere moves and rotates in an arbitrary Stokes flow /1/ (satisfying Stokes's equations). The result was generalized in /2/ to the case of a spherical drop, using the Hadamard-Rybchinskii equation and the reciprocity theorem for the Stokes flows generalized in /3/.

Below, a relatively simple method is presented for determining the forces acting on a spherical particle in an inhomogeneous Stokes flow. The perturbation fields introduced into the flow by the particle are described by a Lamb series /4/. Subsequent integration over the surface of the sphere of the boundary conditions specified on its surface enables us to determine the required integral characteristics in terms of which the force acting on the particle is expressed. The final formulas contain the integrals of the characteristics of the inhomogeneous flow impinging on the sphere, and represent a generalization of the Faxen formulas /1/.

1. When an arbitrary Stokes flow moves past a sphere, a perturbation field described by a Lamb series appears by virtue of the need to satisfy the boundary conditions on the sphere. The force of resistance acting on the sphere is found to depend only on the stresses caused by the presence of the perturbation field. It can be shown that integration of the stresses present in the basic flow over the whole surface of the sphere gives a zero result for any Stokes flow. The contribution of the perturbation field will depend only on the function $p_{-2}$ (the harmonic function appearing in Lamb's solution /4/). The remaining terms of the Lamb expansion make no contributon to the integral expressing the force of resistance $D$, by virtue of the orthogonality of spherical functions of various orders on the sphere. Hence we obtain

$$\mathbf{D} = -3a^{-1}\int_{\Sigma} p_{-2}\mathbf{r}\, ds \tag{1.1}$$

where $a$ is the radius, $\Sigma$ is the surface of the sphere, and $\mathbf{r}$ is the radius vector drawn from the centre of the sphere to a point on its surface.

Let the sphere be situated in an incoming inhomogeneous flow $\mathbf{v}^{\infty}$ determined relative to a frame of reference attached to the sphere. Then, using the boundary conditions on the sphere, we can be obtain a system of equations from which the value of the integral (1.1) can be found.

Let us consider a liquid sphere on whose surface the following conditions must hold: the total radial velocities of the external flow (index $e$) and internal flow (index $i$) $u_r^e = 0$; $u_r^i = 0$ must vanish; the tangential stresses $\mathbf{P}_{r\tau}^e = \mathbf{P}_{r\tau}^i$ are continuous; the total tangential velocities $u_\tau^e = u_\tau^i$ are continuous. The velocity perturbations due to the presence of a spherical drop streamlined by an inhomogeneous flow $\mathbf{v}^{\infty}$ are described by the Lamb series in

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